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Relation between the first and second moments of distributions

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Abstract
A condition on the location of the centre of a mass (or probability) distribution is found if its second moments are given. The result is applied to the relation between the centre of mass and the inertia matrix of bodies. An example is given to illustrate the importance of this condition.

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It is well known that for three given quantities to be the moments of inertia of a real mass distribution with respect to three orthogonal axes intersecting at some point, they must satisfy the triangle inequalities. When one of the triangle inequalities renders an equality, the mass distribution must be planar and its centre of mass should lie in that plane. These facts are mentioned in most textbooks on the mechanics of rigid bodies (see, e.g., [1, 2]). However, the reader is usually left with the impression that the centre of mass of the distribution with given moments of inertia can be chosen in an arbitrary way, which is in fact incorrect.

Let $\Phi(x_1, \ldots, x_n)$ be a normed n-dimensional non-negative (continuous or discrete) distribution function. Denote by $\overline{x_i}$ $(i = 1, \ldots, n)$ the first moments of the distribution with respect to the origin. We have the following:

Theorem 1. *If the matrix of the second moments of the distribution with respect to the origin*

$$S = (s_{ij} = \overline{x_i x_j})_{i,j=1}^{n} \tag{1}$$

is given, then its centre $(\overline{x_1}, \ldots, \overline{x_n})$ *lies inside or on the ellipsoid*

$$\begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ x_1 & s_{11} & s_{12} & \cdots & s_{1n} \\ x_2 & s_{12} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & s_{1n} & s_{2n} & \cdots & s_{nn} \end{vmatrix} = 0 \tag{2}$$

whose principal axes coincide with the eigenvectors of **S** *and whose semi-axes are the square roots of the eigenvalues of* **S**.

Proof. Consider the quadratic form

$$f(\alpha_0, \alpha_1, \dots, \alpha_n) = \overline{(\alpha_0 + \alpha_1 x_1 + \dots + \alpha_n x_n)^2}$$
$$= \alpha_0^2 + 2\alpha_0 \sum_{i=1}^{n} \alpha_i \overline{x_i} + \sum_{i,j=1}^{n} \alpha_i \alpha_j s_{ij}$$
$$= \boldsymbol{\alpha} \mathbf{A} \boldsymbol{\alpha}' \tag{3}$$

where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n)$, α' its transpose and $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & \overline{x_1} & \overline{x_2} & \cdots & \overline{x_n} \\ \overline{x_1} & s_{11} & s_{12} & \cdots & s_{1n} \\ \overline{x_2} & s_{12} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{x_n} & s_{1n} & s_{2n} & \cdots & s_{nn} \end{pmatrix}. \tag{4}$$

The quadratic form (3) is non-negative for all real $\{\alpha_i\}$, and hence $\mathbf{A}$ must have non-negative principal minors [5]:

$$A\begin{pmatrix} i_1 & \cdots & i_m \\ i_1 & \cdots & i_m \end{pmatrix} \geqslant 0 \qquad 1 \leqslant i_1 < \cdots < i_m \leqslant n+1 \quad 1 \leqslant m \leqslant n+1. \tag{5}$$

Turning the axes $x_1, \dots, x_n$ to coincide with the principal axes $\xi_1, \dots, \xi_n$ of $\mathbf{S}$, we replace $\mathbf{A}$ by

$$\begin{pmatrix} 1 & \overline{\xi_1} & \overline{\xi_2} & \cdots & \overline{\xi_n} \\ \overline{\xi_1} & s_{11} & 0 & \cdots & 0 \\ \overline{\xi_2} & 0 & s_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\xi_n} & 0 & 0 & \cdots & s_{nn} \end{pmatrix} \tag{6}$$

and conditions (5) take the form

$$\overline{\xi_i^2} \geqslant 0, \qquad \xi_{i_1}^2 \cdots \xi_{i_m}^2 \left(1 - \frac{\overline{\xi_{i_1}}^2}{\overline{\xi_{i_1}^2}} - \cdots - \frac{\overline{\xi_{i_m}}^2}{\overline{\xi_{i_m}^2}}\right) \geqslant 0. \tag{7}$$

Now we have two cases:

1. If $\mathbf{S}$ is non-singular, then the set of conditions (7) has its intersection as

$$\sum_{i=1}^{n} \frac{\overline{\xi_i}^2}{\overline{\xi_i^2}} \leqslant 1 \tag{8}$$

meaning that the centre of the distribution must lie inside or on the ellipsoid with semi-axes $\left\{\sqrt{\overline{\xi_i^2}}\right\}$ which are directed along the $\xi_1, \dots, \xi_n$ axes. Returning to the axes $\{x_i\}$ the equation of the ellipsoid takes the form stated in the theorem.

Condition (8) is not known in the literature. A fact well known for distributions and widely used in textbooks on statistics and quantum mechanics is that for each variable $\overline{\xi_i}^2 \leqslant \overline{\xi_i^2}$, so that

$$\frac{\overline{\xi_i}^2}{\overline{\xi_i^2}} \leqslant 1 \qquad i = 1, \dots, n. \tag{9}$$

This condition is much weaker than (8) and it signifies that the centre of the distribution lies inside or on the cuboid with sides $\left\{2\sqrt{\overline{\xi_i^2}}\right\}$ formed by tangent planes to the ellipsoid in (8)

Table 1. v_e/v_c for some values of n.

n	1	2	3	4	5	10	20
v_e/v_c	1	0.7854	0.5236	0.3084	0.1645	0.0025	2.5×10^{-8}

at the ends of its axes. As a measure for comparison, let us find the ratio of the volumes v_e, v_c of admissible regions for the centre under conditions (8) and (9), respectively

$$\frac{v_e}{v_c} = \frac{\frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})} \prod_{i=1}^n \sqrt{\overline{\xi_i^2}}}{2^n \prod_{i=1}^n \sqrt{\overline{\xi_i^2}}} = \frac{\left(\frac{\sqrt{\pi}}{2}\right)^n}{\Gamma\left(1+\frac{n}{2}\right)}. \tag{10}$$

This ratio is <1 for all $n > 1$ and decays quickly with growing n. Some values are given in table 1.

2. If $\mathbf{S}$ is singular, then some of its eigenvalues are zeros, say $\overline{\xi_{k+1}^2} = \cdots = \overline{\xi_n^2} = 0$. The part of the conditions (7) in which those quantities appear reduces to

$$\overline{-\xi_{k+1}^2} \geqslant 0, \ldots, -\overline{\xi_n^2} \geqslant 0$$

and hence

$$\overline{\xi_{k+1}} = \cdots = \overline{\xi_n} = 0. \tag{11}$$

The rest of the conditions (7) have the intersection

$$\sum_{i=1}^{k} \frac{\overline{\xi_i}^2}{\overline{\xi_i^2}} \leqslant 1. \tag{12}$$

The centre of the distribution in this case lies inside or on an ellipsoid in the k-dimensional subspace $\xi_{k+1} = \cdots = \xi_n = 0$. □

Application to the centre of mass of a body with given inertia matrix

Let $Oxyz$ be the Cartesian coordinate system coinciding with the principal axes of inertia of a certain body of mass M and given principal moments of inertia A, B and C, respectively. According to the above theorem, the centre of mass of the body lies inside or on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{13}$$

where

$$a^2 = \frac{B+C-A}{2M} \qquad b^2 = \frac{C+A-B}{2M} \qquad c^2 = \frac{A+B-C}{2M}. \tag{14}$$

Although one can expect such a result to be mentioned somewhere in classical textbooks on the mechanics of rigid bodies such as [1, 2], this is not true. This result is new.

Note that if $C \to A + B$ then $c \to 0$ and the centre of mass lies in the ellipse

$$\frac{x^2}{B/M} + \frac{y^2}{A/M} = 1 \tag{15}$$

of the plane $z = 0$.

Such considerations were utilized in our work [3] in the context of stability analysis of certain periodic motions of a rigid body about a fixed point. In some works on applied mechanics these conditions are usually overlooked, leading to unrealistic choices of parameters of rigid bodies in numerical examples. An example is the work [4], where those parameters were

$$A = 10 \qquad B = 20 \qquad C = 30 \qquad M = 300 \qquad x_0 = 2 \qquad y_0 = 5.$$

It is obvious, according to (15), that the values of x_0, y_0 compatible with the given mass and moments must satisfy the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, where $a = 0.2582$ and $b = 0.1826$. If one insists on keeping the given values of x_0, y_0 and modifying only the mass M, a suitable choice would satisfy $M < 0.3704$.

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